

A CURVATURE FORM FOR PSEUDOCONNECTIONS

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ABSTRACT. We obtain the curvature form $F^\nabla = P \circ d^\nabla \circ \nabla - d^\nabla \circ P \circ \nabla + d^\nabla \circ \nabla \circ P$ for a vector bundle pseudoconnection ∇ , where d^∇ is the exterior derivative associated to ∇ . We use F^∇ to obtain the curvature of ∇ . We also prove that $F^\nabla = 0$ is a necessary (but not sufficient) condition for d^∇ to be a chain complex. Instead we prove that $F^\nabla = 0$ and $d^\nabla \circ d^\nabla \circ \nabla = 0$ are necessary and sufficient conditions for d^∇ to be a *chain 2-complex*, i.e., $d^\nabla \circ d^\nabla \circ d^\nabla = 0$.

1. INTRODUCTION

Let M be a differentiable manifold. Denote by $\Omega^0(M)$ the ring of C^∞ real valued maps in M . Denote by $\chi^\infty(M)$ and $\Omega^k(M)$ respectively the $\Omega^0(M)$ -modules of C^∞ vector fields and k -forms defined on M , $k \geq 0$. Denote by $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ the standard exterior derivation of k -forms of M . We denote by $\text{Hom}(A, B)$ the set of homomorphisms from the modulus A to the modulus B . If ξ is a real smooth vector bundle over M we denote by $\Omega^k(\xi)$ the set of ξ -valued k -forms of M , namely, $\Omega^0(\xi)$ is the $\Omega^0(M)$ -module of smooth sections of ξ and $\Omega^k(\xi) = \Omega^k(M) \otimes \Omega^0(\xi)$ for all $k \geq 1$. If η is another vector bundle over M we denote by $\text{HOM}(\xi, \eta)$ the set of bundle homomorphisms from ξ to η over the identity. Every $P \in \text{HOM}(\xi, \eta)$ induces a homomorphism $P \in \text{Hom}(\Omega^0(\xi), \Omega^0(\eta))$ of $\Omega^0(M)$ modules in the usual way. It also defines a homomorphism $P \in \text{Hom}(\Omega^k(\xi), \Omega^k(\eta))$ for all $k \geq 1$ by setting $P(\omega \otimes s) = \omega \otimes P(s)$ at every generator $\omega \otimes s \in \Omega^k(\xi)$.

A *pseudoconnection* of a vector bundle ξ over M is an \mathbb{R} -linear map $\nabla : \Omega^0(\xi) \rightarrow \Omega^0(\xi)$ for which there is a bundle homomorphism $P \in \text{HOM}(\xi, \xi)$ called the principal homomorphism of ∇ such that the Leibnitz rule below holds:

$$\nabla(f \cdot s) = df \otimes P(s) + f \cdot \nabla(s), \quad \forall(f, s) \in \Omega^0(M) \times \Omega^0(\xi).$$

(See [2].) An *ordinary connection* is a pseudoconnection whose principal homomorphism is the identity. By classical arguments we shall associate to any pseudoconnection ∇ an *exterior derivative*, that is, a sequence of linear maps $d^\nabla : \Omega^k(\xi) \rightarrow \Omega^{k+1}(\xi)$ which reduces to ∇ when $k = 0$ and satisfies a Leibnitz rule. We shall use it to define the curvature form $F^\nabla : \Omega^0(\xi) \rightarrow \Omega^2(\xi)$ as the alternating sum

$$F^\nabla = P \circ d^\nabla \circ \nabla - d^\nabla \circ P \circ \nabla + d^\nabla \circ \nabla \circ P.$$

Notice that F^∇ above reduces to the classical curvature form if ∇ were an ordinary connection. We shall prove that F^∇ is a tensor, that is, $F^\nabla \in \text{Hom}(\Omega^0(\xi), \Omega^2(\xi))$, and explain how to obtain the Abe's curvature [2] from F^∇ . We also prove that $F^\nabla = 0$ is a necessary (but not sufficient) condition for d^∇ to be a chain complex.

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Instead we prove that $F^\nabla = 0$ and $d^\nabla \circ d^\nabla \circ \nabla = 0$ are necessary and sufficient conditions for d^∇ to be a *chain 2-complex*, i.e., $d^\nabla \circ d^\nabla \circ d^\nabla = 0$.

2. RESULTS

Let ξ, η be real smooth vector bundles over a differentiable manifold M . An *O-derivative operator from ξ to η with principal homomorphism $P \in \text{HOM}(\xi, \eta)$* is an \mathbb{R} -linear map $\nabla : \Omega^0(\xi) \rightarrow \Omega^0(\eta)$ satisfying the Leibnitz rule

$$\nabla(f \cdot s) = df \otimes P(s) + f \cdot \nabla(s), \quad \forall (f, s) \in \Omega^0(M) \times \Omega^0(\xi).$$

(See [1].) Notice that a pseudoconnection of ξ is nothing but an *O-derivative operator from ξ to itself*.

As in [1] we denote by $O(\xi, \eta; P)$ the set whose elements are the *O-derivative operators with principal homomorphism P from ξ to η* . We even write $O(\xi; P)$ instead of $O(\xi, \xi; P)$ and define

$$O(\xi, \eta) = \bigcup_P O(\xi, \eta; P) \quad \text{and} \quad O(\xi) = \bigcup_P O(\xi; P).$$

Every $\alpha \in \text{HOM}(\xi, \eta)$ induces an alternating product

$$\wedge_\alpha : \Omega^k(M) \times \Omega^l(\xi) \rightarrow \Omega^{k+l}(\eta)$$

defined at the generators by

$$\beta \wedge_\alpha (\omega \otimes s) = (\beta \wedge \omega) \otimes \alpha(s).$$

If $\eta = \xi$ and α is the identity, then \wedge_α reduced to the ordinary alternating product \wedge ([3]).

Lemma 2.1. *For every $\nabla \in O(\xi, \eta; P)$ there is a unique sequence of linear maps $d^\nabla : \Omega^k(\xi) \rightarrow \Omega^{k+1}(\eta)$, $k \geq 0$, satisfying the following properties:*

(1) *If $k = 0$, then*

$$d^\nabla = \nabla.$$

(2) *If $k, l \geq 0$, $\omega \in \Omega^k(M)$ and $S \in \Omega^l(\xi)$, then*

$$d^\nabla(\omega \wedge S) = d\omega \wedge_P S + (-1)^k \omega \wedge d^\nabla S.$$

Proof. First define the map $D^\nabla : \Omega^k(M) \times \Omega^0(\xi) \rightarrow \Omega^{k+1}(\eta)$ by

$$D^\nabla(\omega, s) = d\omega \otimes P(s) + (-1)^k \omega \wedge \nabla s, \quad \forall (\omega, s) \in \Omega^k(M) \times \Omega^0(\xi).$$

Clearly D^∇ is linear and satisfies

$$D^\nabla(f \cdot \omega, s) = D^\nabla(\omega, f \cdot s)$$

for all $f \in \Omega^0(M)$ and $(\omega, s) \in \Omega^k(M) \times \Omega^0(\xi)$. Therefore D^∇ induces a linear map $d^\nabla : \Omega^k(\xi) \rightarrow \Omega^{k+1}(\eta)$ whose value at the generator $\omega \otimes s$ of $\Omega^k(\xi)$ is given by

$$d^\nabla(\omega \otimes s) = d\omega \otimes P(s) + (-1)^k \omega \wedge \nabla s.$$

It follows that d^∇ and ∇ coincide at the generators (for $k = 0$) by the Leibnitz rule of ∇ . Therefore (1) holds. The proof of (2) follows as in [3]. This ends the proof.

The sequence d^∇ in the lemma above will be referred to as the *exterior derivative* of $\nabla \in O(\xi, \eta)$. Next we state the following definition.

Definition 2.2. Let ∇ be a pseudoconnection with principal homomorphism P on a vector bundle ξ . We define the following maps

$$E^\nabla, F^\nabla : \Omega^0(\xi) \rightarrow \Omega^2(\xi), \quad L^\nabla : \Omega^0(\xi) \rightarrow \Omega^1(\xi) \quad \text{and} \quad G^\nabla : \Omega^0(\xi) \rightarrow \Omega^3(\xi)$$

by

- $E^\nabla = d^\nabla \circ \nabla$;
- $F^\nabla = P \circ d^\nabla \circ \nabla - d^\nabla \circ P \circ \nabla + d^\nabla \circ \nabla \circ P$;
- $L^\nabla = P \circ \nabla - \nabla \circ P$;
- $G^\nabla = d^\nabla \circ d^\nabla \circ \nabla$.

The map F^∇ will be referred to as the curvature form of ∇ .

The maps in the definition above are related by the expressions

$$(1) \quad F^\nabla = P \circ E^\nabla - d^\nabla \circ L^\nabla, \quad G^\nabla = d^\nabla \circ E^\nabla.$$

As already observed the curvature form F^∇ of a pseudoconnection ∇ reduces to the classical curvature form when ∇ is an ordinary connection [3].

Theorem 2.3. If ∇ is a pseudoconnection of ξ , then $F^\nabla \in \text{Hom}(\Omega^0(\xi), \Omega^2(\xi))$ and $L^\nabla \in \text{Hom}(\Omega^0(\xi), \Omega^1(\xi))$.

Proof. It is not difficult to see that $L^\nabla \in \text{Hom}(\Omega^0(\xi), \Omega^1(\xi))$. On the other hand,

$$P(\omega \wedge S) = \omega \wedge_P S, \quad \forall \omega \in \Omega^1(M), \forall S \in \Omega^1(\xi)$$

so

$$E^\nabla(f \cdot s) = df \wedge L^\nabla(s) + f \cdot E^\nabla(s), \quad \forall f \in \Omega^0(M), \forall s \in \Omega^0(\xi).$$

Then, (1) implies

$$\begin{aligned} F^\nabla(f \cdot s) &= P(E^\nabla(f \cdot s)) - d^\nabla(L^\nabla(f \cdot s)) = P(df \wedge L^\nabla(s) + f \cdot E^\nabla(s)) - d^\nabla(f \cdot L^\nabla(s)) \\ &= df \wedge_P L^\nabla(s) - df \wedge_P L^\nabla(s) + f \cdot F^\nabla(s) = f \cdot F^\nabla(s) \end{aligned}$$

$\forall f \in \Omega^0(M), \forall s \in \Omega^0(\xi)$. Therefore $F^\nabla \in \text{Hom}(\Omega^0(\xi), \Omega^2(\xi))$ and we are done. This ends the proof.

Lemma 2.4. If ∇ is a pseudoconnection of a vector bundle ξ and $i \geq 0$, then

$$(2) \quad d^\nabla \circ d^\nabla \circ d^\nabla(\omega \otimes s) = d\omega \wedge F^\nabla(s) + (-1)^i \omega \wedge G^\nabla(s)$$

for every generator $\omega \otimes s \in \Omega^i(\xi)$.

Proof. First notice that

$$\begin{aligned} d^\nabla \circ d^\nabla(\omega \otimes s) &= d^\nabla(d^\nabla(\omega \otimes s)) = d^\nabla(d\omega \otimes P(s) + (-1)^i \omega \wedge \nabla s) = \\ &= d^2\omega \otimes P^2(s) + (-1)^{i+1} d\omega \wedge \nabla P(s) + (-1)^i (d\omega \wedge_P \nabla s + \\ &\quad + (-1)^i \omega \wedge d^\nabla(\nabla s)) = (-1)^i [d\omega \wedge (P\nabla s - \nabla Ps) + (-1)^i \omega \wedge d^\nabla(\nabla s)]. \end{aligned}$$

Therefore

$$d^\nabla \circ d^\nabla(\omega \otimes s) = \omega \wedge E^\nabla(s) + (-1)^i d\omega \wedge L^\nabla(s).$$

Applying d^∇ to this expression we get (2). The proof follows.

As is well known the curvature form F^∇ of an ordinary connection ∇ measures how the exterior derivative d^∇ of ∇ deviates from being a *chain complex*, i.e., $d^\nabla \circ d^\nabla = 0$. Indeed, d^∇ is a chain complex if and only if $F^\nabla = 0$. However, the analogous result for pseudoconnections is false in general by Proposition 2.8 below. Despite we shall obtain a pseudoconnection version of this result based on the following definition.

Definition 2.5. A pseudoconnection ∇ is called:

- (1) strongly flat if $E^\nabla = 0$ and $L^\nabla = 0$,
- (2) weakly flat if $F^\nabla = 0$ and $G^\nabla = 0$.

For ordinary connections one has $F^\nabla = E^\nabla$, $L^\nabla = 0$ and then the notions of flatness above coincide with the classical flatness [3]. The exterior derivative d^∇ of a pseudoconnection ∇ is said to be a *chain 2-complex* if $d^\nabla \circ d^\nabla \circ d^\nabla = 0$. With these definitions we have the following result.

Theorem 2.6. A pseudoconnection ∇ is weakly flat (resp. strongly flat) if and only if d^∇ is a chain 2-complex (resp. chain complex).

Proof. We only prove the result for weakly flat since the proof for strongly flat is analogous.

Fix a pseudoconnection ∇ with principal homomorphism P on a vector bundle ξ . If ∇ is weakly flat then d^∇ is a chain 2-complex by (2) in Lemma 2.4. Conversely, if d^∇ is a chain 2-complex, then both $d^\nabla \circ d^\nabla \circ d^\nabla$ and $d^\nabla \circ d^\nabla \circ \nabla$ vanish hence $\omega \wedge F^\nabla(s) = 0$ for all exact form ω of M and all $s \in \Omega^0(\xi)$ by (2) in Lemma 2.4. Since every form in M is locally a $\Omega^0(M)$ -linear combination of alternating product of exact forms we obtain that $\omega \wedge F^\nabla(s) = 0$ for all k -form ω of M ($k \geq 1$) and all $s \in \Omega^0(\xi)$. From this we obtain that $F^\nabla = 0$ so ∇ is weakly flat. The proof follows.

The following is a direct corollary of the above theorem.

Corollary 2.7. If the exterior derivative d^∇ of a pseudoconnection ∇ is a chain complex, then $F^\nabla = 0$.

The converse of the above corollary is false by the following proposition.

Proposition 2.8. There is a pseudoconnection ∇ with $F^\nabla = 0$ such that d^∇ is not a chain complex.

Proof. Choose a suitable vector bundle ξ over $M = \mathbb{R}^3$, $\Phi_2, \Phi_3 \in \text{Hom}(\xi, \xi)$ such that $\Phi_3 \circ \Phi_2 \neq \Phi_2 \circ \Phi_3$ and three 1-forms $\omega_1, \omega_2, \omega_3 \in \Omega^1(M)$ such that $\omega_1 \wedge \omega_2 \wedge \omega_3$ never vanishes. Define the map $\nabla : \Omega^0(\xi) \rightarrow \Omega^1(\xi)$ by

$$\nabla s = \omega_1 \otimes s + \omega_2 \otimes \Phi_2(s) + \omega_3 \otimes \Phi_3(s).$$

We have that $\nabla \in \text{Hom}(\Omega^0(\xi), \Omega^1(\xi))$ therefore ∇ is a pseudoconnection with zero principal homomorphism so $F^\nabla = 0$. On the other hand, an straightforward computation yields

$$G^\nabla(s) = (\omega_1 \wedge \omega_2 \wedge \omega_3) \otimes (\Phi_3 \circ \Phi_2 - \Phi_2 \circ \Phi_3)(s), \quad \forall s \in \Omega^0(\xi)$$

therefore $G^\nabla \neq 0$ and so ∇ is not weakly flat. Then, d^∇ cannot be a chain complex by Theorem 2.6 since a chain complex is necessarily a chain 2-complex. This ends the proof.

To finish we explain how the Abe's curvature [2] can be obtained from the curvature form F^∇ . For this we need some short definitions (see [3]).

Given a vector bundle ξ over M and k vector fields $X_1, \dots, X_k \in \chi^\infty(M)$ we define the evaluation map $Ev_{X_1, \dots, X_k} : \Omega^k(\xi) \rightarrow \Omega^0(\xi)$ by defining

$$Ev_{X_1, \dots, X_k}(\omega \otimes s) = w(X_1, \dots, X_k) \cdot s$$

at each generator $\omega \otimes s \in \Omega^k(\xi)$. If $\nabla \in O(\xi)$ and $X, Y \in \chi^\infty(M)$ we define $\nabla_X : \Omega^0(\xi) \rightarrow \Omega^0(\xi)$ by

$$\nabla_X s = Ev_X(\nabla s)$$

and $F_{X,Y}^\nabla : \Omega^0(\xi) \rightarrow \Omega^0(\xi)$ by

$$F_{X,Y}^\nabla = Ev_{X,Y}(F^\nabla(s)), \quad \forall s \in \Omega^0(\xi).$$

Theorem 2.9. *If $\nabla \in O(\xi; P)$ then*

$$\begin{aligned} F_{X,Y}^\nabla(s) = & \nabla_X \nabla_Y(Ps) - \nabla_Y \nabla_X(Ps) - \nabla_X P(\nabla_Y s) + P \nabla_X \nabla_Y s + \\ & + \nabla_Y P(\nabla_X s) - P \nabla_Y \nabla_X s - P(\nabla_{[X,Y]} P(s)), \end{aligned}$$

for all $X, Y \in \chi^\infty(M)$ and all $s \in \Omega^0(\xi)$.

Proof. By definition we have

$$\begin{aligned} (3) \quad F_{X,Y}^\nabla(s) = & \\ & Ev_{X,Y}(P(d^\nabla(\nabla s))) - Ev_{X,Y}(d^\nabla(P(\nabla s))) + Ev_{X,Y}(d^\nabla(\nabla(Ps))). \end{aligned}$$

Let us compute the three sumands separated way. First of all if $\omega \otimes s \in \Omega^1(\xi)$ is a generator then

$$(4) \quad Ev_{X,Y}(d^\nabla(\omega \otimes s)) = dw(X, Y) \cdot P(s) - \omega(X) \cdot \nabla_Y s + \omega(Y) \cdot \nabla_X s.$$

Now, as $\nabla s \in \Omega^1(\xi)$ and $\{\omega \otimes s' : (\omega, s') \in \Omega^1(M) \times \Omega^0(\xi)\}$ is a generating set of $\Omega^1(\xi)$ we obtain

$$(5) \quad \nabla s = \sum_{r=1}^k \omega_r \otimes s_r,$$

for some $(\omega_r, s_r) \in \Omega^1(M) \times \Omega^0(\xi)$, $r = 1, \dots, k$. Then (4) implies

$$\begin{aligned} (6) \quad Ev_{X,Y}(d^\nabla(\nabla s)) = & \\ & \sum_{r=1}^k \{dw_r(X, Y) \cdot P(s_r) - \omega_r(X) \cdot \nabla_Y s_r + \omega_r(Y) \cdot \nabla_X s_r\}. \end{aligned}$$

On the other hand, (5) yields

$$\nabla_X s = \sum_{r=1}^k \omega_r(X) \cdot s_r$$

therefore

$$\nabla_Y \nabla_X s = \sum_{r=1}^k \{d[\omega_r(X)](Y) \cdot P(s_r) + \omega_r(Y) \cdot \omega_r(X) \cdot \nabla_Y s_r\}.$$

But $\nabla_{[X,Y]} s = \sum_{r=1}^k \omega_r([X, Y]) \cdot s_r$, so

$$P(\nabla_{[X,Y]} s) = \sum_{r=1}^k \omega_r([X, Y]) \cdot P(s_r)$$

and then

$$(7) \quad Ev_{X,Y}(d^\nabla(\nabla s)) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - P(\nabla_{[X,Y]} s)$$

because of (6). Replacing s by $P(s)$ in (7) we obtain

$$(8) \quad Ev_{X,Y}(d^\nabla(\nabla P(s))) = \nabla_X \nabla_Y P(s) - \nabla_Y \nabla_X P(s) - P(\nabla_{[X,Y]} P(s)).$$

Besides (5) implies

$$P(\nabla s) = \sum_{r=1}^k \omega_r \otimes P(s_r)$$

thus

$$\begin{aligned} Ev_{X,Y}(d^\nabla(P\nabla s)) &= \sum_{r=1}^k Ev_{X,Y}(d^\nabla(\omega_r \otimes P(s_r))) = \\ &= \sum_{r=1}^k [dw_r(X, Y) \cdot P^2(s_r) - \omega_r(X) \nabla_X P(s_r) + \omega_r(Y) \cdot \nabla_X P(s_r)] \end{aligned}$$

and then

$$\nabla_Y P(\nabla_X s) = \sum_{r=1}^k \{d[w_r(X)](Y) \cdot P^2(s_r) + w_r(X) \cdot \nabla_Y P(s_r)\}.$$

As $P^2(\nabla_{[X,Y]}s) = \sum_{r=1}^k \omega_r([X, Y]) \cdot P^2(s_r)$ we obtain

$$(9) \quad Ev_{X,Y}(d^\nabla(P(\nabla s))) = \nabla_X P(\nabla_Y s) - \nabla_Y P(\nabla_X s) - P^2(\nabla_{[X,Y]}s).$$

As the maps P and $Ev_{X,Y}$ commute we can apply P to (7) and use (3), (8), (9) to obtain the result.

Remark 2.10. $F_{X,Y}^\nabla(s)$ in Theorem 2.9 is the curvature $K(\nabla)_{X,Y}(s)$ defined in [2] p. 328.

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